

A NOTE ON THE PROBABILITY OF GENERATING ALTERNATING OR SYMMETRIC GROUPS

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ABSTRACT. We improve on recent estimates for the probability of generating the alternating and symmetric groups A_n and S_n . In particular we find the sharp lower bound, if the probability is given by a quadratic in n^{-1} . This leads to improved bounds on the largest number $h(A_n)$ such that a direct product of $h(A_n)$ copies of A_n can be generated by two elements.

1. INTRODUCTION

For a group $X = S_n$ or A_n , we write $p(X)$ for the probability that two elements of X generate a group that contains A_n . In [1], Dixon proved that $p(S_n) \rightarrow 1$ as $n \rightarrow \infty$. In [2] he sharpened this statement to

$$p(S_n) = 1 - \frac{1}{n} - \frac{1}{n^2} - \frac{4}{n^3} - \frac{23}{n^4} - \frac{171}{n^5} - \frac{1542}{n^6} + O(n^{-7}).$$

For many applications, numerical results are needed, rather than asymptotics. In [5] Maróti and Tamburini proved explicit upper and lower bounds

$$1 - \frac{1}{n} - \frac{13}{n^2} < p(X) \leq 1 - \frac{1}{n} + \frac{2}{3n^2}.$$

In this present note, we find the best possible lower bound of this type, and a close-to-optimal upper bound.

Theorem 1.1. *Let $X = A_n$ or $X = S_n$ with $n \geq 5$. Then*

$$1 - \frac{1}{n} - \frac{8.8}{n^2} \leq p(X) < 1 - \frac{1}{n} - \frac{0.93}{n^2}.$$

Equality holds in the lower bound if and only if $n = 6$.

In fact, for $n \geq 14$, we prove that $1 - \frac{1}{n} - \frac{7.5}{n^2} < p(X) < 1 - \frac{1}{n} - \frac{0.93}{n^2}$. The result for smaller n comes from the values for $p(X)$ in Table 1 (taken from [7, Table 4.1]).

Hall [3] considered the largest number $h(S)$ such that a direct product of $h(S)$ copies of a non-abelian finite simple group S can be generated by two elements, and proved that $h(S) = p(S)|S|/|\text{Out}(S)|$. The function $h(S)$ has received considerable attention recently; we refer the reader to [5] for more discussion and references and to [6] for lower bounds on $h(S)$ for all non-abelian finite simple groups S . The new bounds above yield:

Corollary 1.2. *Let n be an integer with $n \geq 14$. Then*

$$\left(1 - \frac{1}{n} - \frac{7.5}{n^2}\right) \left(\frac{n!}{4}\right) < h(A_n) < \left(1 - \frac{1}{n} - \frac{0.93}{n^2}\right) \left(\frac{n!}{4}\right).$$

Let $m(S)$ denote the minimal index of a proper subgroup of a group S . In [4], it is proved that there exist absolute constants c_1 and c_2 such that $1 - c_1/m(S) < p(S) < 1 - c_2/m(S)$, for all non-abelian finite simple groups S . For $i = 1, 2$, let a_i denote the value of c_i for the family of simple alternating groups.

Corollary 1.3. *For $n \geq 5$,*

$$1 - \frac{2.468}{n} < p(A_n) < 1 - \frac{1}{n},$$

and hence $a_1 \leq 2.468$ and $a_2 \geq 1$.

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2. PROOF OF THEOREM 1.1

Definition 2.1. For $X = A_n$ or S_n we let $p_{\text{intrans}}(X)$ and $p_{\text{trans}}(X)$ be the probability that two elements chosen randomly from X generate a subgroup of an intransitive maximal subgroup of X , or a subgroup of a transitive maximal subgroup of X other than A_n , respectively.

Lemma 2.2. *Let $X = A_n$ or S_n with $n \geq 14$. Then*

$$p_{\text{intrans}}(X) < \frac{1}{n} + \frac{2.7}{n^2}.$$

Proof. We prove the result for S_n , the arguments for A_n are identical. Let $x, y \in S_n$ and suppose that $Y := \langle x, y \rangle$ is contained in an intransitive maximal subgroup. Then Y is contained in a subgroup conjugate to $S_k \times S_{n-k}$ for some $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$.

Let $k \in \{1, \dots, n-1\}$. Then the probability that $Y \leq S_k \times S_{n-k}$ is bounded by

$$\binom{n}{k} \left(\frac{k!(n-k)!}{n!} \right)^2 = \binom{n}{k}^{-1}.$$

So the probability that $Y \leq S_1 \times S_{n-1}$ is at most $\frac{1}{n}$, and the probability that $Y \leq S_2 \times S_{n-2}$ and Y is transitive on the orbit of size 2 is bounded by

$$\frac{3}{4} \frac{2}{n(n-1)} = \frac{3}{2n(n-1)}.$$

Similarly, the probability that $Y \leq S_3 \times S_{n-3}$ and Y is transitive on the orbit of length 3 is

$$\frac{13}{18} \binom{n}{3}^{-1} = \frac{13}{3n(n-1)(n-2)}.$$

Now the probability that $Y \leq S_k \times S_{n-k}$ for some $4 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ is

$$\sum_{k=4}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{\binom{n}{k}} \leq \sum_{k=4}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{\binom{n}{4}} \leq \frac{12(n-7)}{n(n-1)(n-2)(n-3)}.$$

We now observe that, since $n \geq 14$,

$$\frac{3}{2n(n-1)} + \frac{13}{3n(n-1)(n-2)} + \frac{12(n-7)}{n(n-1)(n-2)(n-3)} < \frac{2.7}{n^2}$$

which completes the proof. \square

Lemma 2.3. *Let $X = A_n$ or S_n , with $n \geq 14$. Then*

$$p_{\text{intrans}}(X) > \frac{1}{n} + \frac{0.93}{n^2}.$$

Proof. We observe that $p_{\text{intrans}}(X)$ is bounded below by the probability that a random pair of elements of X generate a subgroup with a fixed point, or with an orbit of size 2. For $X = S_n$, we bound $p_{\text{intrans}}(X)$ by doing inclusion-exclusion to depth 2 on the union of the sets $(S_n)_\alpha$, with $1 \leq \alpha \leq n$, and $(S_n)_{\{\alpha, \beta\}} \setminus (S_n)_{(\alpha, \beta)}$, with $1 \leq \alpha < \beta \leq n$. We find that $p_{\text{intrans}}(X)$ is greater than

$$\frac{1}{n} + \frac{3}{4} \frac{2(n-2)!}{n!} - \frac{(n-2)!}{2n!} - \frac{3}{4} \binom{n}{1} \binom{n-1}{2} \left(\frac{2(n-3)!}{n!} \right)^2 - \left(\frac{3}{4} \right)^2 \frac{\binom{n}{2} \binom{n-2}{2}}{2} \left(\frac{4(n-4)!}{n!} \right)^2$$

Thus

$$p_{\text{intrans}}(X) \geq \frac{1}{n} + \frac{8n^2 - 52n + 75}{8n(n-1)(n-2)(n-3)}$$

which, since $n \geq 14$, is greater than $\frac{1}{n} + \frac{0.93}{n^2}$. \square

Proof of Theorem 1.1. For the upper bound we use Lemma 2.3. For the lower bound, note that

$$1 - p(X) = p_{\text{intrans}}(X) + p_{\text{trans}}(X).$$

It follows from the proofs of [5, Lemmas 3.1 and 4.3] that $p_{\text{trans}}(X) \leq \frac{4.8}{n^2}$. Combining this with Lemma 2.2 gives the theorem. \square

In Table 1 we record the value of $p(A_n)$ and $p(S_n)$ for $n \leq 13$, together with our lower and upper bounds as stated in Theorem 1.1. All values are correct to three decimal places.

TABLE 1. Precise values and bounds for $p(X)$

n	5	6	7	8	9	10	11	12	13
$p(A_n) =$	0.633	0.588	0.726	0.739	0.848	0.875	0.893	0.902	0.913
$p(S_n) =$	0.633	0.588	0.795	0.796	0.859	0.875	0.894	0.903	0.913
$p(X) \geq$	0.448	0.588	0.677	0.737	0.780	0.812	0.836	0.855	0.871
$p(X) \leq$	0.763	0.808	0.839	0.861	0.878	0.891	0.902	0.911	0.918

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